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Twisted Hopf symmetries of canonical noncommutative spacetimes and the no-pure-boost principle

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Abstract

We study the twisted-Hopf-algebra symmetries of observer-independent canonical spacetime noncommutativity, for which the commutators of the spacetime coordinates take the form $[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}$ with observer-independent (and coordinate-independent) $\theta^{\mu\nu}$. We find that it is necessary to introduce nontrivial commutators between transformation parameters and spacetime coordinates, and that the form of these commutators implies that all symmetry transformations must include a translation component. We show that with our noncommutative transformation parameters the Noether analysis of the symmetries is straightforward, and we compare our canonical-noncommutativity results with the structure of the conserved charges and the "no-pure-boost" requirement derived in a previous study of κ-Minkowski noncommutativity. We also verify that, while at intermediate stages of the analysis we do find terms that depend on the ordering convention adopted in setting up the Weyl map, the final result for the conserved charges is reassuringly independent of the choice of Weyl map and (the corresponding choice of) star product.

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I. INTRODUCTION

Over these past few years there has been strong interest in the study of theories formulated in noncommutative versions of the Minkowski spacetime. The most studied possibility is the one of spacetime noncommutativity of "canonical" [1] form

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu} \ , \tag{1}$$

where $\hat{x}^\mu$ are the spacetime coordinates ($\mu \in \{0,1,2,3\}$, time coordinate $\hat{x}^0$) and $\theta^{\mu\nu}$ is coordinate-independent. The literature on this possibility is extremely large, also because the same formula (1) can actually represent rather different physical scenarios, depending on the properties attributed to $\theta^{\mu\nu}$. The earliest studies we are aware of were actually the ones [2] in which richer properties were attributed to $\theta^{\mu\nu}$, including some constraints [2] on the admissible forms of $\theta^{\mu\nu}$ and the possibility of nontrivial algebraic properties [3]. And this picture can find valuable motivation in the outcome of certain heuristic analyses of limitations on the localization of a spacetime point in the quantum-gravity realm [2]. The formalism is much simpler if one analyzes (1) assuming that $\theta^{\mu\nu}$ is a (dimensionful) number-valued tensor [4, 5, 6], and this gives rise to a picture which could be rather valuable, since it is believed to provide an accurate effective-theory description of String Theory in presence of a certain tensor background [4, 5, 6]. The tensor background breaks the spacetime symmetries in just the way codified by the tensor $\theta^{\mu\nu}$: the laws of physics are different in different frames because $\theta^{\mu\nu}$ (transforming like a Lorentz-Poincaré tensor) takes different values in different frames. The third possibility is for $\theta^{\mu\nu}$ to be a number-valued observer-independent matrix. This would of course require the laws of transformation between inertial observers to be modified in θ-dependent manner [7, 8, 9]. Preliminary results [10, 11, 12] suggest that this might be accomplished by introducing a description of translations, boosts and space-rotation transformations based on the formalism of Hopf algebras.

We intend to focus here on this third possibility, looking for a deeper understanding of the structure of the Hopf-algebra symmetry transformations and hoping to set the stage for a more physical characterization of this novel concept. In particular, we are interested in establishing similarities and differences between the Hopf-algebra symmetries of canonical spacetimes and the Hopf-algebra symmetries of the so-called κ-Minkowski spacetime, for which some of us recently reported a Noether analysis [13, 14].

The key ingredient which allowed [13, 14] the completion (after more than a decade of failed attempts) of some Noether analyses in the κ-Minkowski case is the introduction of "noncommutative transformation parameters" with appropriate nontrivial commutators with the spacetime coordinates. And interestingly the form of the commutators between transformation parameters and spacetime coordinates turns out to be incompatible with the possibility of a pure boost.

We intend to show here that analogous structures appear in the analysis of the Hopf-algebra symmetries of observer-independent canonical spacetime noncommutativity. In this case we find that neither a pure boost nor a pure rotation are allowed, and, combining these results with the ones previously obtained for κ-Minkowski, we conjecture a (limited) universality of a no-pure-boost uncertainty principle for Hopf-algebra symmetries of noncommutative Minkowski-like spacetimes.

We also stress the significance of the fact that our Noether analysis derives 10 conserved charges from the Poincaré like Hopf-algebra symmetries. This provides encouragement for the idea that these Hopf-algebra symmetries are truly meaningful in characterizing observable aspects of the relevant theories, contrary to what feared by some authors (see, *e.g.*, Ref. [15]), who had argued that the Hopf-algebra structures encountered in the study of canonical noncommutative spacetimes might be just a fancy mathematical formalization of a rather trivial break down of symmetry.

Guided again by intuition developed in our previous studies of κ-Minkowski [13, 14, 16] we also expose an ordering issue for the so-called classical-action description of the generators of symmetry transformations in canonical noncommutative spacetime. While this issue should be carefully monitored in future analyses of other aspects of theories in canonical noncommutative spacetimes, we reassuringly find that our result for the charges has no dependence on this choice of ordering prescription.

II. TWISTED-HOPF SYMMETRY ALGEBRA AND ORDERING ISSUES

Our first task is to show that the much studied [10, 11, 12] "twisted" Hopf algebra of (candidate) symmetries of canonical noncommutative spacetime can be obtained by introducing rules of "classical action" [16] for the generators of the symmetry algebra. We start by observing that the fields one considers in constructing theories in a canonical noncommutative spacetime can be written in the form [1]:

$$\Phi(\hat{x}) = \int d^4k\, \tilde{\Phi}_w(k) e^{ik\hat{x}} \tag{2}$$

by introducing ordinary (commutative) "Fourier parameters" k_μ [1].

This associates to any given function $\Phi(\hat{x})$ a "Fourier transform" $\tilde{\Phi}_w(k)$, and it is customary to take this one step further by using this as the basis for an association, codified in a "Weyl map" Ω_w, between the noncommutative functions $\Phi(\hat{x})$ of interest and some auxiliary commutative functions $\Phi_w^{(comm)}(x)$:

$$\Phi(\hat{x}) = \Omega_w\left(\Phi_w^{(comm)}(x)\right) \equiv \Omega_w\left(\int d^4k\, \tilde{\Phi}_w(k) e^{ikx}\right) = \int d^4k\, \tilde{\Phi}_w(k) e^{ik\hat{x}} \tag{3}$$

It is easy to verify that this definition of the Weyl map Ω_w acts on a given commutative function by giving a noncommutative function with full symmetrization ("Weyl ordering") on the noncommutative spacetime coordinates (*e.g.*, $\Omega_w(e^{ikx}) = e^{ik\hat{x}}$ and $\Omega_w(x_1 x_2^2) = \frac{1}{3}\left(\hat{x}_2^2\hat{x}_1 + \hat{x}_2\hat{x}_1\hat{x}_2 + \hat{x}_1\hat{x}_2^2\right)$).

We shall stress that it is also legitimate to consider Weyl maps with other ordering prescriptions, but before we do that let us first use Ω_w for our description of the relevant twisted Hopf algebra. This comes about by introducing rules of "classical action" for the generators of translations and space rotations and boosts:[2]:

$$P_\mu^{(w)} e^{ik\hat{x}} \equiv P_\mu^{(w)}\Omega_w(e^{ikx}) \equiv \Omega_w(P_\mu e^{ikx}) = \Omega_w(i\partial_\mu e^{ikx}) \tag{4}$$

$$M_{\mu\nu}^{(w)} e^{ik\hat{x}} \equiv M_{\mu\nu}^{(w)}\Omega_w(e^{ikx}) \equiv \Omega_w(M_{\mu\nu} e^{ikx}) = \Omega_w(ix_{[\mu}\partial_{\nu]} e^{ikx})\ . \tag{5}$$

Here the antisymmetric "Lorentz-sector" matrix of operators $M_{\mu\nu}^{(w)}$ is composed as usual by the space-rotation generators $R_i^{(w)} = \frac{1}{2}\epsilon_{ijk}M_{jk}^{(w)}$ and the boost generators $N_i^{(w)} = M_{0i}^{(w)}$. The rules of action codified in (4)-(5) are said to be "classical actions according to the Weyl map Ω_w" since they indeed reproduce the corresponding classical rules of action within the Weyl map.

It is easy to verify that the generators introduced in (4)-(5) satisfy the same commutation relations of the classical Poincaré algebra:

$$\begin{aligned}
\left[P_\mu^{(w)}, P_\nu^{(w)}\right] &= 0 \\
\left[P_\alpha^{(w)}, M_{\mu\nu}^{(w)}\right] &= i\eta_{\alpha[\mu}P_{\nu]}^{(w)} \\
\left[M_{\mu\nu}^{(w)}, M_{\alpha\beta}^{(w)}\right] &= i\left(\eta_{\alpha[\nu}M_{\mu]\beta}^{(w)} + \eta_{\beta[\mu}M_{\nu]\alpha}^{(w)}\right)\ .
\end{aligned} \tag{6}$$

However, the action of Lorentz-sector generators does not comply with Leibniz rule,

$$\begin{aligned}
M_{\mu\nu}^{(w)}\left(e^{ik\hat{x}}e^{iq\hat{x}}\right) &= \left(M_{\mu\nu}^{(w)}e^{ik\hat{x}}\right)e^{iq\hat{x}} + e^{ik\hat{x}}\left(M_{\mu\nu}^{(w)}e^{iq\hat{x}}\right) + \\
&-\frac{1}{2}\theta^{\alpha\beta}\left[\eta_{\alpha[\mu}\left(P_{\nu]}^{(w)}e^{ik\hat{x}}\right)\left(P_\beta^{(w)}e^{iq\hat{x}}\right) + \left(P_\alpha^{(w)}e^{ik\hat{x}}\right)\eta_{\beta[\mu}\left(P_{\nu]}^{(w)}e^{iq\hat{x}}\right)\right]\ ,
\end{aligned} \tag{7}$$

---

[1] We use $\hat{x}$ for noncommuting coordinates, x for the auxiliary commuting ones.

[2] In light of (2) one obtains a fully general rule of action of operators by specifying their action only on the exponentials $e^{ik\hat{x}}$. Also note that we adopt a standard compact notation for antisymmetrized indices: $A_{[\alpha\beta]} \equiv A_{\alpha\beta} - A_{\beta\alpha}$.

as one easily verifies using the fact that from (1) it follows that

$$e^{ik\hat{x}}e^{iq\hat{x}} = e^{i(k+q)\hat{x}}e^{-\frac{i}{2}k^\mu\theta_{\mu\nu}q^\nu} \equiv \Omega_w(e^{i(k+q)x}e^{-\frac{i}{2}k^\mu\theta_{\mu\nu}q^\nu}). \tag{8}$$

For the translation generators instead Leibniz rule is satisfied,

$$P^{(w)}_\mu\left(e^{ik\hat{x}}e^{iq\hat{x}}\right) = \left(P^{(w)}_\mu e^{ik\hat{x}}\right)e^{iq\hat{x}} + e^{ik\hat{x}}\left(P^{(w)}_\mu e^{iq\hat{x}}\right) , \tag{9}$$

as one could have expected from the form of the commutators (1), which is evidently compatible with classical translation symmetry (while, for observer-independent $\theta^{\mu\nu}$, it clearly requires an adaptation of the Lorentz sector.)

In the relevant literature observations of the type reported in (7) and (9) are often described via an Hopf algebraic structure, specifying the coproduct

$$\begin{aligned}\Delta P^{(w)}_\mu &= P^{(w)}_\mu \otimes \mathbb{1} + \mathbb{1} \otimes P^{(w)}_\mu , \\ \Delta M^{(w)}_{\mu\nu} &= M^{(w)}_{\mu\nu} \otimes \mathbb{1} + \mathbb{1} \otimes M^{(w)}_{\mu\nu} - \frac{1}{2}\theta^{\alpha\beta}\left[\eta_{\alpha[\mu}P^{(w)}_{\nu]} \otimes P^{(w)}_\beta + P^{(w)}_\alpha \otimes \eta_{\beta[\mu}P^{(w)}_{\nu]}\right] .\end{aligned} \tag{10}$$

Antipode and counit, the other two building blocks needed for a Hopf algebra, can also be straightforwardly introduced [17], but do not play a role in the analysis we are here reporting.

It turns out that the coproducts (10) are describable as a deformation of the classical Poincaré Lie algebra by the following twist element:

$$\mathcal{F} = e^{\frac{i}{2}\theta^{\mu\nu}P^{(w)}_\mu \otimes P^{(w)}_\nu} . \tag{11}$$

The form of the twist element is most easily obtained from the structure of the star product, which is a way to reproduce the rule of product of noncommutative functions within the Weyl map: $e^{ik\hat{x}}e^{iq\hat{x}} \equiv \Omega_w(e^{ikx} \star e^{iqx})$. From (8) we see that our star product must be such that $e^{ikx} \star e^{iqx} = e^{i(k+q)x}e^{-\frac{i}{2}k^\mu\theta_{\mu\nu}q^\nu}$, and denoting by $\bar{\mathcal{F}} \equiv \sum(\bar{f}_1 \otimes \bar{f}_2)$ the representation of the inverse of the twist element $\mathcal{F}^{-1}$ on $\mathcal{A} \otimes \mathcal{A}$ (where $\mathcal{A}$ is the algebra of commutative functions $f(x)$) we must have [10] that $\Omega_w(g(x) * h(x)) = \Omega_w(\sum(\bar{f}_1(g))(\bar{f}_2(h)))$, from which (11) follows.

Hopf algebras that are obtained from a given Lie algebra by exclusively acting with a twist element preserve the form of the commutators among generators, so that all the structure of the deformation is codified in the coproducts. And these coproducts are structured in such a way that for a generator G_θ, obtained twisting G, the coproduct Δ_θ has the form $\Delta_\theta(G_\theta) = \mathcal{F}\Delta(G)\mathcal{F}^{-1}$.

Having established that by introducing "classical action according to Ω_w" for translations, space-rotations and boosts one obtains a certain set of generators for a twisted Hopf algebra, it is natural to ask if something different is encountered if these generators are introduced with classical action according to a different Weyl map, such as the Weyl map Ω_1 defined by $\Omega_1(e^{ikx}) = e^{ik^A\hat{x}_A}e^{ik^1\hat{x}_1}$, where $A = 0, 2, 3$.

A given filed $\Phi(\hat{x})$ which according to the Weyl map Ω_w has Fourier transform $\tilde{\Phi}_w(k)$ (in the sense of (2)), according to Ω_1 has a different Fourier transform $\Phi_1(k)$,

$$\Phi(\hat{x}) = \int d^4k\tilde{\Phi}_1(k)e^{ik^A\hat{x}_A}e^{ik^1\hat{x}_1}, \tag{12}$$

and, since $e^{ik\hat{x}} = e^{ik^A\hat{x}_A}e^{ik^1\hat{x}_1}e^{\frac{i}{2}k^Ak^1\theta_{A1}}$, the two Fourier transforms are simply related:

$$\tilde{\Phi}_1(k) = \tilde{\Phi}_w(k)e^{-\frac{i}{2}k^Ak^1\theta_{A1}} . \tag{13}$$

Denoting by $P^{(1)}_\mu$ and $M^{(1)}_{\mu\nu}$ the generators with "classical action according to Ω_1" one easily finds that they also leave invariant the commutation relations (1). And, as most easily verified [17]

through a simple analysis of the action of these generators on $e^{ik\hat{x}} = \Omega_w(e^{ikx}) = \Omega_1(e^{ikx}e^{\frac{i}{2}k^A k^1 \theta_{A1}})$, the following relations hold:

$$
\begin{aligned}
P^{(1)}_\mu &= P^{(w)}_\mu \equiv P_\mu \ , \\
M^{(1)}_{\mu\nu} &= M^{(w)}_{\mu\nu} + \frac{1}{2}\theta^{A1}[\eta_{1[\mu}P_{\nu]}P_A + \eta_{A[\mu}P_{\nu]}P_1] \ .
\end{aligned} \tag{14}
$$

Setting aside the difference between $M^{(1)}_{\mu\nu}$ and $M^{(w)}_{\mu\nu}$, one could say that the construction based on the two Weyl maps Ω_w and Ω_1 lead to completely analogous structures. Again one easily uncovers the structure of a twisted Hopf algebra, the commutators of generators are undeformed, and all the structure of the deformation is in a coproduct relation, which in the case of the Ω_1 map takes the form

$$
\begin{aligned}
\Delta M^{(1)}_{\mu\nu} &= M^{(1)}_{\mu\nu} \otimes \mathbb{1} + \mathbb{1} \otimes M^{(1)}_{\mu\nu} - \frac{1}{2}\theta^{\alpha\beta}\left[\eta_{\alpha[\mu}P_{\nu]} \otimes P_\beta + P_\alpha \otimes \eta_{\beta[\mu}P_{\nu]}\right] + \\
&+ \frac{1}{2}\theta^{A1}\left[\eta_{A[\mu}P_{\nu]} \otimes P_1 + \eta_{1[\mu}P_{\nu]} \otimes P_A + P_1 \otimes P_{[\nu}\eta_{\mu]A} + P_A \otimes P_{[\nu}\eta_{\mu]1}\right] \ .
\end{aligned} \tag{15}
$$

This may be viewed again as the result of "twisting", which in this case would be due to the following twist element

$$
\mathcal{F}_1 = e^{\frac{i}{2}\theta_{AB}P^A \otimes P^B} e^{-i\theta_{A1}P^1 \otimes P^A}, \tag{16}
$$

where $A, B = 0, 2, 3$.

The two sets of generators $\{P_\mu, M^{(1)}_{\mu\nu}\}$ and $\{P_\mu, M^{(w)}_{\mu\nu}\}$ can be meaningfully described as two bases of generators for the same twisted Hopf algebra. However, we shall keep track of the structures we encounter as a result of the difference between $M^{(1)}_{\mu\nu}$ and $M^{(w)}_{\mu\nu}$, which, since these differences merely amount to a choice of ordering convention, we expect not to affect the observable features of our theory.

III. NONCOMMUTATIVE TRANSFORMATION PARAMETERS

Our analysis of canonical noncommutativity will be guided by the description of symmetry transformations for κ-Minkowski spacetime noncommutativity reported by some of us in Refs. [13, 14]. After the failures of several other attempts, the criteria adopted in Refs. [13, 14] finally allowed us to complete successfully the Noether analysis, including the identification of some conserved (time-independent) charges associated to the symmetries. We shall therefore assume that those criteria should be also adopted in the case of canonical noncommutativity.

In Refs. [13, 14] κ-Poincaré symmetry transformations of a function $f(\hat{x})$ of the κ-Minkowski spacetime coordinates were parametrized as follows:

$$
df(\hat{x}) = i\left(\gamma^\mu P_\mu + \sigma_j R_j + \tau_k N_k\right) f(\hat{x}), \tag{17}
$$

where γ_μ, σ_j, τ_k are the transformation parameters (respectively translation, space-rotation and boost parameters), and P_μ, R_j, N_k are, respectively, translation, space-rotation and boost generators.

The properties of the transformation parameters γ_μ, σ_j and τ_k were derived [13, 14] by imposing Leibniz rule on the d,

$$
d(f(\hat{x})g(\hat{x})) = (df(\hat{x}))g(\hat{x}) + f(\hat{x})(dg(\hat{x})) \ . \tag{18}
$$

It turned out that this requirement cannot be satisfied by standard (commutative) transformation parameters, so Refs. [13, 14] introduced the concept of "noncommutative transformation parameters" as the most conservative generalization of the standard concept of transformation parameters

that would allow to satisfy the Leibniz rule. These noncommutative transformation parameters were required to still act only by (associative) multiplication on the spacetime coordinates, but were allowed to be subject to nontrivial rules of commutation with the spacetime coordinates. An intriguing aspect of the commutators between transformation parameters and spacetime coordinates derived in Refs. [13, 14] is that they turn out to be incompatible with the possibility of a pure boost. The structure of κ-Minkowski spacetime does allow pure translations and pure space-rotations, but when the boost parameters are not set to zero then also the space-rotation parameters must not all be zero.

We intend to introduce here an analogous description of the twisted Hopf symmetry transformations of canonical spacetimes. We consider first the case of the generators $P_\mu, M_{\mu\nu}^{(w)}$, with classical action according to the Weyl map Ω_w, and we start by analyzing the case of a pure translation transformation:

$$d_P f(\hat{x}) = i\gamma^\mu_{(w)} P_\mu f(\hat{x}). \tag{19}$$

Imposing Leibniz rule, because of the triviality of the coproduct of the translation generators (see previous section), for this case of a pure translation transformation one easily verifies that the condition imposed by compliance with Leibniz rule,

$$[f(\hat{x}), \gamma^\mu_{(w)}] P_\mu g(\hat{x}) = 0 \tag{20}$$

is also trivial and is satisfied by ordinary commutative transformation parameters.

For the case of a pure Lorentz-sector transformation,

$$d_L f(\hat{x}) = i\omega^{\mu\nu}_{(w)} M^{(w)}_{\mu\nu} f(\hat{x}), \tag{21}$$

by imposing Leibniz rule one arrives at the following nontrivial requirement:

$$[f(\hat{x}), \omega^{\mu\nu}_{(w)}] M^{(w)}_{\mu\nu} g(\hat{x}) = -\frac{1}{2}\omega^{\mu\nu}_{(w)} (\theta_{[\mu}{}^{\sigma}\delta_{\nu]}{}^{\rho} + \theta^{\rho}{}_{[\mu}\delta_{\nu]}{}^{\sigma})(P_\rho f(\hat{x}))(P_\sigma g(\hat{x})) \ . \tag{22}$$

This does not admit any solution of the type we are allowing for the transformation parameters. In fact, in order to be solutions of (22) the $\omega^{\mu\nu}_{(w)}$ should be operators with highly nontrivial action on functions of the spacetime coordinates, rather than being "noncommutative parameters", acting by simple (associative) multiplication on the spacetime coordinates.

We conclude that whereas pure translations are allowed in canonical spacetimes, the possibility of a pure Lorentz-sector transformation is excluded.

We find however that, while pure Lorentz-sector transformations are not allowed, it is possible to combine Lorentz-sector and translation transformations. In fact, if we consider a transformation with

$$df(\hat{x}) = i\left[\gamma^\alpha_{(w)} P_\alpha + \omega^{\mu\nu}_{(w)} M^{(w)}_{\mu\nu}\right] f(\hat{x}) \ , \tag{23}$$

then the Leibniz-rule requirement takes the form

$$\left[[f(\hat{x}), \gamma^\alpha_{(w)}] + \frac{1}{2}\omega^{\mu\nu}_{(w)} (\theta_{[\mu}{}^{\alpha}\delta_{\nu]}{}^{\rho} + \theta^{\rho}{}_{[\mu}\delta_{\nu]}{}^{\alpha}) \left(P_\rho f(\hat{x})\right)\right] P_\alpha g(\hat{x}) + [f(\hat{x}), \omega^{\mu\nu}_{(w)}] M^{(w)}_{\mu\nu} g(\hat{x}) = 0 \ , \tag{24}$$

which amounts (by imposing that the term proportional to $P_\alpha g(\hat{x})$ and the term proportional to $M^{(w)}_{\mu\nu} g(\hat{x})$ be separately null) to the following requirements

$$\begin{aligned}\left[f(\hat{x}), \gamma^\alpha_{(w)}\right] &= -\frac{1}{2}\omega^{\mu\nu}_{(w)} (\theta_{[\mu}{}^{\alpha}\delta_{\nu]}{}^{\rho} + \theta^{\rho}{}_{[\mu}\delta_{\nu]}{}^{\alpha}) P_\rho f(\hat{x}) \\ \left[f(\hat{x}), \omega^{\mu\nu}_{(w)}\right] &= 0 \ .\end{aligned} \tag{25}$$

And these requirements imply the following properties of the transformation parameters

$$\left[\hat{x}^\beta, \gamma^\alpha_{(w)}\right] = -\frac{i}{2}\omega^{\mu\nu}_{(w)}(\theta_{[\mu}{}^\alpha \delta_{\nu]}{}^\beta + \theta^\beta{}_{[\mu}\delta_{\nu]}{}^\alpha) \quad (26)$$

$$\left[\hat{x}^\beta, \omega^{\mu\nu}_{(w)}\right] = 0 \ , \quad (27)$$

which are consistent with our criterion for noncommutative transformation parameters, since they introduce indeed a noncommutativity between transformation parameters and spacetime coordinates, but in a way that is compatible with our requirement that the transformation parameters act only by (associative) multiplication on the spacetime coordinates.

We conclude that Lorentz-sector transformations are allowed but only in combination with translation transformations. Indeed (26) is such that whenever $\omega_{(w)} \neq 0$ then also $\gamma_{(w)} \neq 0$. And interestingly the translation-transformation parameters, which can be commutative in the case of a pure translation transformation, must comply with (26), and therefore be noncommutative parameters, in the general case of a transformation that combines a translation component and a Lorentz-sector component.

Since in the preceding section we raised the issue of possible alternatives to the $P_\mu, M^{(w)}_{\mu\nu}$ basis, such as the basis $P_\mu, M^{(1)}_{\mu\nu}$ obtained by a different ordering prescription in the Weyl map used to introduce the "classical action" of the generators, we should stress here that the analysis of transformation parameters proceeds in exactly the same way if one works with the basis $P_\mu, M^{(1)}_{\mu\nu}$; however, the noncommutativity properties of the transformation parameters are somewhat different. In the case $P_\mu, M^{(1)}_{\mu\nu}$ one ends up considering transformations of the form

$$d^{(1)}f(\hat{x}) = i\left[\gamma^\alpha_{(1)}P_\alpha + \omega^{\mu\nu}_{(1)}M^{(1)}_{\mu\nu}\right]f(\hat{x}) \ , \quad (28)$$

and it is easy to verify that the transformation parameters must satisfy the following noncommutativity requirements:

$$\left[f(\hat{x}), \gamma^\alpha_{(1)}\right] = -\frac{1}{2}\omega^{\mu\nu}_{(1)}\Upsilon^{\alpha\rho}_{\mu\nu}P_\rho f(\hat{x})$$

$$\left[f(\hat{x}), \omega^{\mu\nu}_{(1)}\right] = 0, \quad (29)$$

where $\Upsilon^{\alpha\rho}_{\mu\nu} = (\theta_{[\mu}{}^\alpha\delta_{\nu]}{}^\rho + \theta^\rho{}_{[\mu}\delta_{\nu]}{}^\alpha) - \theta^{A1}[\eta_{A[\mu}\delta_{\nu]}{}^\rho\delta_1{}^\alpha + \eta_{1[\mu}\delta_{\nu]}{}^\rho\delta_A{}^\alpha + \eta_{A[\mu}\delta_{\nu]}{}^\alpha\delta_1{}^\rho + \eta_{1[\mu}\delta_{\nu]}{}^\alpha\delta_A{}^\rho]$.

We shall show that, even though the differences between $P_\mu, M^{(w)}_{\mu\nu}$ and $P_\mu, M^{(1)}_{\mu\nu}$ require different forms of the commutators between transformation parameters and spacetime coordinates, these two possible choices of convention for the description of the symmetry Hopf algebra lead to the same conserved charges.

IV. CONSERVED CHARGES

We now test our formulation of twisted-Hopf-algebra symmetry transformations in the context of a Noether analysis of the simplest and most studied theory formulated in canonical noncommutative spacetime: a theory for a massless scalar field $\phi(\hat{x})$ governed by the following Klein-Gordon-like equation of motion:

$$\Box\phi(\hat{x}) \equiv P_\mu P^\mu \phi(\hat{x}) = 0. \quad (30)$$

Consistently with the analysis reported in the previous section, we want to obtain conserved charges associated to the transformations of the form

$$\delta\phi(\hat{x}) = -d\phi(\hat{x}) = -i\left[\gamma^\alpha_{(w)}P_\alpha + \omega^{\mu\nu}_{(w)}M^{(w)}_{\mu\nu}\right]\phi(\hat{x}) \ , \quad (31)$$

where the first equality holds because the field we are considering is a scalar.

We take as starting point for the Noether analysis the action

$$S=\frac{1}{2}\int d^4\hat{x}\,\phi(\hat{x})\Box\phi(\hat{x}), \tag{32}$$

which (as one can easily verify [17]) generates the equation of motion (30) and is invariant under the transformation (31):

$$\begin{aligned}\delta S &= \frac{1}{2}\int d^4\hat{x}\,(\delta\phi(\hat{x})\Box\phi(\hat{x})+\phi(\hat{x})\Box\delta\phi(\hat{x})-d(\phi(\hat{x})\Box\phi(\hat{x})))= \\ &= \frac{1}{2}\int d^4\hat{x}\phi(\hat{x})[\Box,\delta]\phi(\hat{x})=0.\end{aligned} \tag{33}$$

Of course, the charges are to be obtained for fields solutions of the equation of motion, and therefore we can use (30) to rewrite (33) in the following way:

$$\delta S=\frac{1}{2}\int d^4\hat{x}\,\phi(\hat{x})\Box\delta\phi(\hat{x})=\frac{1}{2}\int d^4\hat{x}\,P_\mu\left[\phi(\hat{x})P^\mu\delta\phi(\hat{x})-(P^\mu\phi(\hat{x}))\delta\phi(\hat{x})\right]. \tag{34}$$

Then using the commutation relations of the infinitesimal parameters obtained in Eq. (25) one can further rewrite δS in the following insightful manner:

$$\delta S=-i\int d^4\hat{x}\left(\gamma_\nu^{(w)}P_\mu T^{\mu\nu}+\omega_{(w)}^{\rho\sigma}P_\mu J^\mu_{\rho\sigma}\right), \tag{35}$$

with

$$\begin{aligned}T^{\mu\nu} &= \frac{1}{2}\left(\phi(\hat{x})P^\mu P^\nu\phi(\hat{x})-(P^\mu\phi(\hat{x}))P^\nu\phi(\hat{x})\right), \\ J^\mu_{\rho\sigma} &= \frac{1}{2}\left(\phi(\hat{x})P^\mu M^{(w)}_{\rho\sigma}\phi(\hat{x})-(P^\mu\phi(\hat{x}))M^{(w)}_{\rho\sigma}\phi(\hat{x})\right)+ \\ & -\frac{1}{4}(\theta_{[\rho}{}^\nu\delta_{\sigma]}{}^\lambda+\theta^\lambda{}_{[\rho}\delta_{\sigma]}{}^\nu)\left[(P_\lambda\phi(\hat{x}))P^\mu P_\nu\phi(\hat{x})-(P^\mu P_\lambda\phi(\hat{x}))P_\nu\phi(\hat{x})\right].\end{aligned} \tag{36}$$

It is rather easy to verify that by spatial integration of the 0-th components of the "currents" $T^{\mu\nu}$ and $J^\mu_{\rho\sigma}$ one obtains time-independent charges. Denoting this charges with Q_μ,$K_{\rho\sigma}$,

$$Q_\mu=\int d^3\hat{x}\,T^0_\mu,\qquad K_{\rho\sigma}=\int d^3\hat{x}\,J^0_{\rho\sigma}\ , \tag{37}$$

and using the ordering convention (2) for the Fourier expansion of a generic field which is solution of the equation of motion,

$$\phi(\hat{x})=\int d^4k\,\delta(k^2)\tilde{\phi}_{(w)}(k)e^{ik\hat{x}}\ , \tag{38}$$

upon integration over the spatial coordinates[3] one finds:

$$\begin{aligned}Q_\mu &= \frac{1}{2}\int d^4k\,d^4q\,\delta(k^2)\delta(q^2)\tilde{\phi}_{(w)}(k)\tilde{\phi}_{(w)}(q) \\ & \left(q^0-k^0\right)q_\mu\delta^{(3)}(\vec{k}+\vec{q})e^{i(k^0+q^0)\hat{x}_0}e^{\frac{i}{2}(k^0+q^0)(k^i+q^i)\theta_{i0}}e^{-\frac{i}{2}k^\mu q^\nu\theta_{\mu\nu}}\ ,\end{aligned} \tag{39}$$

[3] Our spatial Dirac deltas are such that $\int d^3\hat{x}e^{ik^i\hat{x}_i}=\delta^{(3)}(\vec{k})$.

$$K_{\rho\sigma} = \frac{1}{2}\int d^4k\, d^4q\, \delta(k^2)\tilde{\phi}_{(w)}(k)\left[iq_{[\rho}\frac{\partial}{\partial q^{\sigma]}}[\delta(q^2)\tilde{\phi}_{(w)}(q)] - \frac{1}{2}\delta(q^2)(\theta_{[\rho}{}^{\nu}\delta_{\sigma]}{}^{\lambda} + \theta^{\lambda}{}_{[\rho}\delta_{\sigma]}{}^{\nu})k_\lambda q_\nu \tilde{\phi}_{(w)}(q)\right]\cdot$$
$$\cdot\left(k^0 - q^0\right)\delta^{(3)}(\vec{k}+\vec{q})e^{i(k^0+q^0)\hat{x}_0}e^{\frac{i}{2}(k^0+q^0)(k^i+q^i)\theta_{i0}}e^{-\frac{i}{2}k^\mu q^\nu\theta_{\mu\nu}}\ . \qquad (40)$$

Then integrating in d^4k, and observing that in $K_{\rho\sigma}$ the term $-\frac{1}{2}(\theta_{[\rho}{}^{\nu}\delta_{\sigma]}{}^{\lambda} + \theta^{\lambda}{}_{[\rho}\delta_{\sigma]}{}^{\nu})k_\lambda q_\nu \tilde{\phi}(q)$ gives null contribution, one obtains:

$$Q_\mu = \frac{1}{2}\int \frac{d^4q}{2|\vec{q}|}\,\delta(q^2)\tilde{\phi}_{(w)}(q)q_\mu\left\{\tilde{\phi}_{(w)}(-\vec{q},|\vec{q}|)\left(q^0+|\vec{q}|\right)e^{i(q^0-|\vec{q}|)\hat{x}_0}e^{-\frac{i}{2}(q^0-|\vec{q}|)q^i\theta_{0i}} + \right.$$
$$\left. +\tilde{\phi}_{(w)}(-\vec{q},-|\vec{q}|)\left(q^0-|\vec{q}|\right)e^{i(q^0+|\vec{q}|)\hat{x}_0}e^{-\frac{i}{2}(q^0+|\vec{q}|)q^i\theta_{0i}}\right\}, \qquad (41)$$

$$K_{\rho\sigma} = \frac{i}{2}\int \frac{d^4q}{2|\vec{q}|}\,\delta(q^2)\tilde{\phi}_{(w)}(q)q_{[\rho}\left\{(q^0+|\vec{q}|)\left[\frac{\partial}{\partial q^{\sigma]}}\tilde{\phi}_{(w)}(-\vec{q},|\vec{q}|)\right]e^{i(q^0-|\vec{q}|)\hat{x}_0}e^{-\frac{i}{2}(q^0-|\vec{q}|)q^i\theta_{0i}} + \right.$$
$$\left. +(q^0-|\vec{q}|)\left[\frac{\partial}{\partial q^{\sigma]}}\tilde{\phi}_{(w)}(-\vec{q},-|\vec{q}|)\right]e^{i(q^0+|\vec{q}|)\hat{x}_0}e^{-\frac{i}{2}(q^0+|\vec{q}|)q^i\theta_{0i}}\right\}. \qquad (42)$$

One can then use the fact that $\delta(q^2)$ imposes $q_0 = \pm|\vec{q}|$, and the presence of factors of the types $(q^0-|\vec{q}|)e^{\alpha(q^0+|\vec{q}|)}$ and $(q^0+|\vec{q}|)e^{\alpha(q^0-|\vec{q}|)}$ to obtain the following explicitly time-independent formulas for the charges:

$$Q_\mu = \frac{1}{2}\int \frac{d^4q}{2|\vec{q}|}\,\delta(q^2)\tilde{\phi}_{(w)}(q)q_\mu\left\{\tilde{\phi}_{(w)}(-\vec{q},|\vec{q}|)\left(q^0+|\vec{q}|\right) + \tilde{\phi}_{(w)}(-\vec{q},-|\vec{q}|)\left(q^0-|\vec{q}|\right)\right\}, \qquad (43)$$

$$K_{\rho\sigma} = \frac{i}{2}\int \frac{d^4q}{2|\vec{q}|}\,\delta(q^2)\tilde{\phi}_{(w)}(q)q_{[\rho}\left\{(q^0+|\vec{q}|)\frac{\partial\tilde{\phi}_{(w)}(-\vec{q},|\vec{q}|)}{\partial q^{\sigma]}} + (q^0-|\vec{q}|)\frac{\partial\tilde{\phi}_{(w)}(-\vec{q},-|\vec{q}|)}{\partial q^{\sigma]}}\right\}. \qquad (44)$$

V. ORDERING-CONVENTION INDEPENDENCE OF THE CHARGES

In light of the "choice-of-ordering issue" we raised in Section II, which in particular led us to consider the examples of two possible bases of generators, the $P_\mu, M^{(w)}_{\mu\nu}$ basis and the $P_\mu, M^{(1)}_{\mu\nu}$ basis, and especially considering the fact that in Section III we found that in different bases the noncommutative transformation parameters should have somewhat different properties (different form of the commutators with the spacetime coordinates), it is interesting to verify whether or not the result for the charges obtained in the previous section working with the $P_\mu, M^{(w)}_{\mu\nu}$ basis is confirmed by a corresponding analysis based on the $P_\mu, M^{(1)}_{\mu\nu}$ basis.

When adopting the $P_\mu, M^{(1)}_{\mu\nu}$ basis the symmetry variation of a field is described by

$$\delta\phi(\hat{x}) = -d^{(1)}\phi(\hat{x}) = -i\left[\gamma^\alpha_{(1)}P_\alpha + \omega^{\mu\nu}_{(1)}M^{(1)}_{\mu\nu}\right]\phi(\hat{x}), \qquad (45)$$

rather than (31). And going through the same type of steps discussed in the previous section the analysis of the symmetry variation of the action (32) then leads to [17] the following formulas for the currents:

$$T^{\mu\nu\,(1)} = \frac{1}{2}\left(\phi(\hat{x})P^\mu P^\nu\phi(\hat{x}) - P^\mu\phi(\hat{x})P^\nu\phi(\hat{x})\right)\ , \qquad (46)$$

$$J^{\mu\,(1)}_{\rho\sigma} = \frac{1}{2}\left(\phi(\hat{x})P^\mu M^{(1)}_{\rho\sigma}\phi(\hat{x}) - P^\mu\phi(\hat{x})M^{(1)}_{\rho\sigma}\phi(\hat{x})\right) +$$
$$-\frac{1}{4}\Upsilon^{\nu\lambda}_{\rho\sigma}\left[P_\lambda\phi(\hat{x})P^\mu P_\nu\phi(\hat{x}) - P^\mu P_\lambda\phi(\hat{x})P_\nu\phi(\hat{x})\right]\ , \qquad (47)$$

where we used again the compact notation $\Upsilon^{\nu\lambda}_{\rho\sigma}$, introduced in Section III.

The current $T^{\mu\nu\,(1)}$ is manifestly equal to the current $T^{\mu\nu}$ obtained in the previous section using the $P_\mu, M^{(w)}_{\mu\nu}$ basis. Therefore the corresponding charges also coincide:

$$Q^{(1)}_\mu \equiv \int d^3\hat{x}\; T^{0\,(1)}_\mu = \int d^3\hat{x}\; T^0_\mu = Q_\mu. \tag{48}$$

The current $J^{\mu\,(1)}_{\rho\sigma}$ does differ from $J^\mu_{\rho\sigma}$ of the previous section in two ways: in place of the factor $\Upsilon^{\nu\lambda}_{\rho\sigma}$ of $J^{\mu\,(1)}_{\rho\sigma}$ one finds in $J^\mu_{\rho\sigma}$ the factor $\theta_{[\rho}{}^{\nu}\delta_{\sigma]}{}^{\lambda} + \theta^\lambda{}_{[\rho}\delta_{\sigma]}{}^{\nu}$, and there are two (operator) factors $M^{(1)}_{\mu\nu}$ in places where in $J^\mu_{\rho\sigma}$ one of course has $M^{(w)}_{\mu\nu}$. Still, once again the final result for the charges is unaffected:

$$K^{(1)}_{\rho\sigma} \equiv \int d^3\hat{x}\; J^{0\,(1)}_{\rho\sigma} = K_{\rho\sigma}. \tag{49}$$

This is conveniently verified by following the ordering conventions of Eq. (12) in writing the generic solution of the equation of motion,

$$\phi(\hat{x}) = \int d^4k\; \delta(k^2)\tilde{\phi}_{(1)}(k) e^{ik^A\hat{x}_A} e^{ik^1\hat{x}_1}\;, \tag{50}$$

thereby obtaining, after spatial integration, the following formula for $K^{(1)}_{\rho\sigma}$:

$$\begin{aligned} K^{(1)}_{\rho\sigma} \;=\; & \frac{1}{2}\int d^4k\, d^4q\, \delta(k^2)\tilde{\phi}_{(1)}(k)\left[iq_{[\rho}\frac{\partial}{\partial q^{\sigma]}}[\delta(q^2)\tilde{\phi}_{(1)}(q)] - \frac{1}{2}\delta(q^2)\Upsilon^{\nu\lambda}_{\rho\sigma}k_\lambda q_\nu \tilde{\phi}_{(1)}(q)\right]\cdot \\ & \cdot\left(k^0-q^0\right)\delta^{(3)}(\vec{k}+\vec{q}) e^{i(k^0+q^0)\hat{x}_0} e^{\frac{i}{2}(k^0+q^0)(k^i+q^i)\theta_{i0}} e^{-\frac{i}{2}k^\mu q^\nu\theta_{\mu\nu}} e^{-\frac{i}{2}(k^Ak^1+q^Aq^1)\theta_{A1}}\;. \end{aligned} \tag{51}$$

And, using observations that are completely analogous to some we discussed in the previous section, one easily manages [17] to rewrite $K^{(1)}_{\rho\sigma}$ as follows:

$$\begin{aligned} K^{(1)}_{\rho\sigma} \;=\; & \frac{i}{2}\int \frac{d^4q}{2|\vec{q}|}\,\tilde{\phi}_{(1)}(q)\delta(q^2)q_{[\rho}\left\{ (q^0+|\vec{q}|)\frac{\partial}{\partial q^{\sigma]}}\left[\tilde{\phi}_{(1)}(-\vec{q},|\vec{q}|)e^{-i\left(q^A\delta^j_A q^1\theta_{j1}+\frac{1}{2}(|\vec{q}|+q^0)\theta_{01}\right)}\right]\right. \\ & \left. +(q^0-|\vec{q}|)\frac{\partial}{\partial q^{\sigma]}}\left[\tilde{\phi}_{(1)}(-\vec{q},-|\vec{q}|)e^{-i\left(q^A\delta^j_A q^1\theta_{j1}+\frac{1}{2}(-|\vec{q}|+q^0)\theta_{01}\right)}\right]\right\}\;. \end{aligned} \tag{52}$$

This formula for $K^{(1)}_{\rho\sigma}$ is easily shown to reproduce the corresponding formula for $K_{\rho\sigma}$, using the fact that, as we showed in Section III, $\tilde{\phi}_{(1)}(k) = \tilde{\phi}_{(w)}(k)e^{-\frac{i}{2}k^Ak^1\theta_{A1}}$.

This result establishes that the values of the charges carried by a given noncommutative field can be treated as objective facts, independent of the choice of ordering prescription adopted in the analysis. Working with different ordering prescriptions one arrives at different formulas (for example (44) and (52)) expressing the charges as functionals of the Fourier transform of the fields, but these differences in the formulas are just such to compensate for the differences between the Fourier transforms of a given field that are found adopting different ordering conventions, and therefore the values of the charges carried by a given noncommutative field can be stated in an ordering-prescription-independent manner.

VI. CLOSING REMARKS

The fact that we managed to derive a full set of 10 conserved charges from the twisted-Hopf-algebra symmetries that emerge from observer-independent canonical noncommutativity certainly provides some encouragement for the idea that these (contrary to some expectations formulated in

the recent literature [15]) are genuine physical symmetries. And this viewpoint is strengthened by our result on the ordering-convention independence of the charges.

The characterization of "noncommutative transformation parameters" introduced by some of us in Refs. [13, 14], for the analysis of theories in κ-Minkowski noncommutative spacetime, proved to be valuable also in the present study of canonical noncommutativity. This type of transformation parameters objectively does the job (without any need of "further intervention") of allowing to derive conserved charges, but it requires still some work for what concerns establishing its physical implications and its realm of applicability. Is this only an appropriate recipe for deriving conserved charges? Or can we attribute to it all the roles that transformation parameters have in a classical-spacetime theory? For example: does the noncommutativity of these parameters imply that the concept of angle of rotation around a given axis is "fuzzy" in a canonical spacetime?

The obstruction encountered in the previous analyses [14] of κ-Minkowski spacetime for the realization of a pure boost reappeared here in the analysis of canonical spacetimes (actually accompanied by an additional obstruction for the realization of a pure space rotation). Since, to our acknowledge, canonical and κ-Minkowski spacetimes are the only examples of noncommutative versions of Minkowski spacetime that one can single out with some reasonable physical criteria (see, *e.g.*, Ref. [18] and references therein), the fact that in both cases pure boosts are not allowed could perhaps motivate the search of an intuitive argument for the emergence of a universal "no-pure-boost principle" from the general structure of spacetime noncommutativity.

A lot remains to be done for a proper characterization of the physical/observable implications of canonical noncommutativity. By which measurements can a theory with observer-independent canonical noncommutativity be distinguished from a corresponding classical-spacetime theory? Of course, this issue would be most naturally addressed in the context of a theory of quantum fields in the noncommutative spacetime, which we have postponed to future work. But even within analyses of classical fields in canonical spacetime, such as the one we here reported, a preliminary investigation of "observability issues" could be attempted. Correspondingly new measurement-procedure ideas are needed in order to test the novel possibility of an obstruction for the realization of a pure Lorentz-sector transformation. And more work is also needed for a proper operative characterization of the differences between the charges here obtained for a theory with observer-independent canonical noncommutativity and the corresponding charges of a theory in classical spacetime.

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